

On the p-adic valuation of harmonic numbers

*Original*

On the p-adic valuation of harmonic numbers / Sanna, Carlo. - In: JOURNAL OF NUMBER THEORY. - ISSN 0022-314X. - STAMPA. - 166:(2016), pp. 41-46. [10.1016/j.jnt.2016.02.020]

*Availability:*

This version is available at: 11583/2722661 since: 2020-05-03T10:02:48Z

*Publisher:*

Academic Press Incorporated

*Published*

DOI:10.1016/j.jnt.2016.02.020

*Terms of use:*

openAccess

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

*Publisher copyright*

(Article begins on next page)

# ON THE $p$ -ADIC VALUATION OF HARMONIC NUMBERS

CARLO SANNA

ABSTRACT. For any prime number  $p$ , let  $J_p$  be the set of positive integers  $n$  such that  $p$  divides the numerator of the  $n$ -th harmonic number  $H_n$ . An old conjecture of Eswarathasan and Levine states that  $J_p$  is finite. We prove that for  $x \geq 1$  the number of integers in  $J_p \cap [1, x]$  is less than  $129p^{2/3}x^{0.765}$ . In particular,  $J_p$  has asymptotic density zero. Furthermore, we show that there exists a subset  $S_p$  of the positive integers, with logarithmic density greater than 0.273, and such that for any  $n \in S_p$  the  $p$ -adic valuation of  $H_n$  is equal to  $-\lfloor \log_p n \rfloor$ .

## 1. INTRODUCTION

For each positive integer  $n$ , let

$$H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

be the  $n$ -th harmonic number. The arithmetic properties of harmonic numbers have been studied since a long time. For example, Wolstenholme [7] proved in 1862 that for any prime number  $p \geq 5$  the numerator of  $H_{p-1}$  is divisible by  $p^2$ ; while in 1915, Taisinger [6, p. 3115] showed that  $H_n$  is never an integer for  $n > 1$ .

For each prime number  $p$ , let  $J_p$  be the set of positive integers  $n$  such that the numerator of  $H_n$  is divisible by  $p$ . Eswarathasan and Levine [4] conjectured that  $J_p$  is finite for all primes  $p$ , and provided a method to compute the elements of  $J_p$ . If  $J_p$  is finite, then, after sufficient computation, their method gives a proof that it is finite. They computed  $J_2 = \emptyset$ ,  $J_3 = \{2, 7, 22\}$ ,  $J_5 = \{4, 20, 24\}$ , and

$$J_7 = \{6, 42, 48, 295, 299, 337, 341, 2096, 2390, 14675, 16731, 16735, 102728\}.$$

Boyd [2], using some  $p$ -adic expansions, improved the algorithm of Eswarathasan and Levine, and determined  $J_p$  for all primes  $p \leq 547$ , but 83, 127, and 397; confirming that  $J_p$  is finite for those prime numbers. Notably, he showed that  $J_{11}$  has 638 elements, the largest being an integer of 31 digits. Boyd gave also an heuristic model predicting that  $J_p$  is always finite and that its cardinality is  $\#J_p = O(p^2(\log \log p)^{2+\varepsilon})$ . However, the conjecture of Eswarathasan and Levine is still open.

We write  $J_p(x) := J_p \cap [1, x]$ , for  $x \geq 1$ . Our first result is the following.

**Theorem 1.1.** *For any prime number  $p$  and any  $x \geq 1$ , it holds*

$$\#J_p(x) < 129p^{2/3}x^{0.765}.$$

*In particular,  $J_p$  has asymptotic density zero.*

For any prime number  $p$ , let  $\nu_p(\cdot)$  be the usual  $p$ -adic valuation over the rational numbers. Boyd [2, Proposition 3.3] proved the following lemma.

**Lemma 1.2.** *For any prime  $p$ , the set  $J_p$  is finite if and only if  $\nu_p(H_n) \rightarrow -\infty$ , as  $n \rightarrow +\infty$ .*

Therefore, the study of  $J_p$  is strictly related to the negative growth of the  $p$ -adic valuation of  $H_n$ . It is well-known and easy to prove that  $\nu_2(H_n) = -\lfloor \log_2 n \rfloor$ . (Hereafter,  $\lfloor x \rfloor$  denotes the greatest integer not exceeding the real number  $x$ .) Moreover, Kamano [5, Theorem 2] proved

---

2010 *Mathematics Subject Classification.* Primary: 11B99, 11B83. Secondary: 11B05.

*Key words and phrases.* harmonic numbers,  $p$ -adic valuation, asymptotic density, logarithmic density.

that  $\nu_3(H_n)$  can be determined easily from the expansion of  $n$  in base 3. Note that, since obviously  $\nu_p(k) \leq \lfloor \log_p n \rfloor$  for any  $k \in \{1, \dots, n\}$ , we have the lower bound

$$(1) \quad \nu_p(H_n) \geq -\lfloor \log_p n \rfloor.$$

Our next result shows that in (1) the equality holds quite often. We recall that the logarithmic density of a set of positive integers  $S$  is defined as

$$\delta(S) := \lim_{x \rightarrow +\infty} \frac{1}{\log x} \sum_{n \in S \cap [1, x]} \frac{1}{n},$$

whenever this limit exists.

**Theorem 1.3.** *For any prime number  $p$ , there exists a set  $S_p$  of positive integers, with logarithmic density  $\delta(S_p) > 0.273$ , and such that  $\nu_p(H_n) = -\lfloor \log_p n \rfloor$  for each  $n \in S_p$ .*

## 2. PROOF OF THEOREM 1.1

For any prime  $p$ , define the sequence of sets  $J_p^{(1)}, J_p^{(2)}, \dots$  as follow:

$$\begin{aligned} J_p^{(1)} &:= \{n \in \{1, \dots, p-1\} : p \mid H_n\}, \\ J_p^{(k+1)} &:= \{pn + r : n \in J_p^{(k)}, r \in \{0, \dots, p-1\}, p \mid H_{pn+r}\} \quad \forall k \geq 1. \end{aligned}$$

First, we need the following lemma.

**Lemma 2.1.** *For all prime numbers  $p$ , it holds  $J_p^{(k)} = J_p \cap [p^{k-1}, p^k[$ , for each integer  $k \geq 1$ . In particular,  $J_p = \bigcup_{k=1}^{\infty} J_p^{(k)}$ .*

*Proof.* From [4, Eq. 2.5] we know that if  $n$  is a positive integer and  $r \in \{0, \dots, p-1\}$ , then  $pn + r \in J_p$  implies that  $n \in J_p$ . Therefore, the claim follows quickly by induction on  $k$ .  $\square$

Now we prove a result regarding the number of elements of  $J_p$  in a short interval.

**Lemma 2.2.** *For any prime  $p$ , and any real numbers  $x$  and  $y$ , with  $1 \leq y < p$ , we have*

$$\#(J_p \cap [x, x+y]) < \frac{3y^{2/3}}{2} + 1.$$

*Proof.* Set  $c := \#(J_p \cap [x, x+y])$ . If  $c \leq 1$ , then there is nothing to prove. Hence, suppose  $c \geq 2$  and let  $n_1 < \dots < n_c$  be the elements of  $J_p \cap [x, x+y]$ . Moreover, define  $d_i := n_{i+1} - n_i$ , for any  $i = 1, \dots, c-1$ . Given a positive integer  $d$ , consider the polynomial

$$(2) \quad f_d(X) := (X+1)(X+2) \cdots (X+d).$$

Taking the logarithms of both sides of (2) and deriving, we obtain the identity

$$\frac{f'_d(X)}{f_d(X)} = \frac{1}{X+1} + \frac{1}{X+2} + \cdots + \frac{1}{X+d}.$$

Thus for any  $i = 1, \dots, c-1$  we have

$$\frac{f'_{d_i}(n_i)}{f_{d_i}(n_i)} = \frac{1}{n_i+1} + \frac{1}{n_i+2} + \cdots + \frac{1}{n_{i+1}} = H_{n_{i+1}} - H_{n_i} \equiv 0 \pmod{p},$$

so that  $f'_{d_i}(n_i) \equiv 0 \pmod{p}$ . Since  $f'_d(X)$  is a non-zero polynomial of degree  $d-1$ , there are at most  $d-1$  solutions modulo  $p$  of the equation  $f'_d(X) \equiv 0 \pmod{p}$ . Therefore, for any  $z \geq 1$ , on the one hand we have

$$(3) \quad \#\{i : d_i \leq z\} = \sum_{1 \leq d \leq z} \#\{i : d_i = d\} \leq \sum_{1 \leq d \leq z} (d-1) < \frac{z^2}{2}.$$

On the other hand,

$$(4) \quad \#\{i : d_i > z\} < \frac{1}{z} \sum_{i=1}^{c-1} d_i = \frac{n_c - n_1}{z} \leq \frac{y}{z}$$

In conclusion, by summing (3) and (4), we get

$$c - 1 = \#\{i : d_i \leq z\} + \#\{i : d_i > z\} < \frac{z^2}{2} + \frac{y}{z}$$

and the claim follows taking  $z = y^{1/3}$ .  $\square$

We are ready to prove Theorem 1.1. If  $p < 83$  then, from the values of  $\#J_p$  computed by Boyd [2, Table 2], one can check that  $\#J_p/p^{2/3} < 129$ , so the claim is obvious. Hence, suppose  $p \geq 83$ , and put  $A := \frac{3}{2}(p-1)^{2/3} + 1$ . By the definition of the sets  $J_p^{(k)}$ , and since Lemma 2.2, we get that

$$\#J_p^{(1)} = \#(J_p \cap [1, p-1]) < A,$$

while

$$\#J_p^{(k+1)} = \sum_{n \in J_p^{(k)}} \#(J_p \cap [pn, pn+p-1]) < \#J_p^{(k)} \cdot A,$$

hence it follows by induction that  $\#J_p^{(k)} < A^k$ .

Now let  $s$  be the positive integer determined by  $p^{s-1} \leq x < p^s$ . Note that  $p^s \notin J_p$ , indeed  $\nu_p(H_{p^s}) = -s$  (this is a particular case of Lemma 3.2 in the next section). Thanks to Lemma 2.1 and the previous considerations, we have

$$\begin{aligned} \#J_p(x) &\leq \#J_p(p^s) = \#J_p(p^s - 1) = \sum_{k=1}^s \#(J_p \cap [p^{k-1}, p^k]) = \sum_{k=1}^s \#J_p^{(k)} \\ &< \sum_{k=1}^s A^k < \frac{A^2}{A-1} \cdot A^{s-1} = \frac{A^2}{A-1} \cdot (p^{s-1})^{\log_p A} < \frac{A^2}{A-1} \cdot x^{0.765} < 129p^{2/3}x^{0.765}, \end{aligned}$$

since  $p^{s-1} \leq x$ , while it can be checked quickly that  $\log_p A < 0.765$ . The proof is complete.

### 3. PROOF OF THEOREM 1.3

For any integer  $b \geq 2$  and any  $d \in \{1, \dots, b-1\}$ , let  $F_b(d)$  be the set of positive integers that have the most significant digit of their base  $b$  expansion equal to  $d$ . The set  $F_b(d)$  has not an asymptotic density, however  $F_b(d)$  has a logarithmic density. In fact,  $F_b(d)$  satisfies a kind of Benford's law [1], as shown by the following lemma.

**Lemma 3.1.** *For all integers  $b \geq 2$  and  $d \in \{1, \dots, b-1\}$ , it holds  $\delta(F_b(d)) = \log_b(1 + 1/d)$ .*

*Proof.* See [3].  $\square$

Write  $J_p^* := \{1, \dots, p-1\} \setminus J_p^{(1)}$ .

**Lemma 3.2.** *For  $p$  prime,  $d \in J_p^*$ , and  $n \in F_p(d)$ , it holds  $\nu_p(H_n) = -\lfloor \log_p n \rfloor$ .*

*Proof.* Since  $n \in F_p(d)$ , we can write  $n = p^k d + r$ , where  $k := \lfloor \log_p n \rfloor$  and  $r < p^k$  is a non-negative integer. Hence,

$$(5) \quad H_n = \sum_{\substack{m=1 \\ p^k \nmid m}}^n \frac{1}{m} + \sum_{j=1}^d \frac{1}{p^k j} = \sum_{\substack{m=1 \\ p^k \nmid m}}^n \frac{1}{m} + \frac{H_d}{p^k}.$$

On the one hand, it is clear that the last sum in (5) has  $p$ -adic valuation greater than  $-k$ . On the other hand, we have  $\nu_p(H_d/p^k) = -k$ , since  $d \in J_p^*$  and so  $p \nmid H_d$ .

In conclusion,  $\nu_p(H_n) = -k$  as desired.  $\square$

Now we can prove Theorem 1.3. Define the set  $S_p$  as

$$S_p := \bigcup_{d \in J_p^*} F_p(d).$$

It follows immediately from Lemma 3.2 that  $\nu_p(H_n) = -\lfloor \log_p n \rfloor$ , for each  $n \in S_p$ . Moreover, since the sets  $F_p(d)$  are disjoint, and thanks to Lemma 3.1, we have

$$(6) \quad \begin{aligned} \delta(S_p) &:= \sum_{d \in J_p^*} \delta(F_p(d)) = \sum_{d \in J_p^*} \log_p \left( 1 + \frac{1}{d} \right) \geq \sum_{d = \#J_p^{(1)} + 1}^{p-1} \log_p \left( 1 + \frac{1}{d} \right) \\ &= \log_p \left( \frac{p}{\#J_p^{(1)} + 1} \right) = 1 - \frac{\log(\#J_p^{(1)} + 1)}{\log p}. \end{aligned}$$

Suppose  $p \geq 1013$ . By Lemma 2.2 we have

$$\#J_p^{(1)} = \#(J_p \cap [1, p-1]) < \frac{3}{2}(p-2)^{2/3} + 1,$$

hence from (6) we get

$$\delta(S_p) > 1 - \frac{\log(\frac{3}{2}(p-2)^{2/3} + 2)}{\log p} > 0.273.$$

At this point, the proof is only a matter of computation. The author used the Python programming language (since it has native support for arbitrary-sized integers) to compute the numerators of the harmonic numbers  $H_n$ , up to  $n = 1012$ . Then he determined  $\#J_p^{(1)}$  for each prime number  $p < 1013$ , and using (6) he checked that the inequality  $\delta(S_p) > 0.273$  holds. This required only a few seconds on a personal computer.

#### 4. CONCLUDING REMARKS

From the proof of Theorem 1.1, it is clear that with our methods one cannot obtain an upper bound better than  $\#J_p(x) < Cp^{2/3}x^{2/3+\varepsilon}$ , for some  $C, \varepsilon > 0$ . Similarly, in the statement of Theorem 1.3 a logarithmic density greater than  $1/3 - \varepsilon$  cannot be achieved.

One way to obtain better results could be an improvement of Lemma 2.2, we leave this as an open question for the readers.

#### REFERENCES

1. A. Berger and T. P. Hill, *An introduction to Benford's law*, Princeton University Press, 2015.
2. D. W. Boyd, *A p-adic study of the partial sums of the harmonic series*, Experiment. Math. **3** (1994), no. 4, 287–302.
3. R. L. Duncan, *Note on the initial digit problem*, Fibonacci Quart. **7** (1969), no. 5, 474–475.
4. A. Eswarathasan and E. Levine, *p-integral harmonic sums*, Discrete Math. **91** (1991), no. 3, 249–257.
5. K. Kamano, *On 3-adic valuations of generalized harmonic numbers*, Integers **11** (2011).
6. E. W. Weisstein, *Concise encyclopedia of mathematics*, 2 ed., CRC Press, 2002.
7. J. Wolstenholme, *On certain properties of prime numbers*, Quart. J. Pure Appl. Math. **5** (1862), 35–39.

UNIVERSITÀ DEGLI STUDI DI TORINO, DEPARTMENT OF MATHEMATICS, TURIN, ITALY  
E-mail address: `carlo.sanna.dev@gmail.com`